THE MACROSCOPIC APPROACH TO EXTENDED THERMODYNAMICS WITH 14 MOMENTS, UP TO WHATEVER ORDER

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Abstract: Extended Thermodynamics is the natural framework in which to study the physics of fluids, because it leads to symmetric hyperbolic systems of field laws, thus assuming important properties such as finite propagation speeds of shock waves and well posedness of the Cauchy problem. The closure of the system of balance equations is obtained by imposing the entropy principle and that of galilean relativity. If we take the components of the mean field as independent variables, these two principles are equivalent to some conditions on the entropy density and its flux. The method until now used to exploit these conditions, with the macroscopic approach, has not been used up to whatever order with respect to thermodynamical equilibrium. This is because it leads to several difficulties in calculations. Now these can be overcome by using a new method proposed recently by Pennisi and Ruggeri. Here we apply it to the 14 moments model. We will also show that the 13 moments case can be obtained from the present one by using the method of subsystems.
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1. Introduction

The 14 moments model was firstly investigated by Kremer [1], up to second order with respect to equilibrium; here we want to exploit it up to whatever order. The appropriate balance equations for this model reads

$$\begin{aligned}
&\partial_t F + \partial_k F^k = 0 \\
&\partial_t F^i + \partial_k F^{ik} = 0 \\
&\partial_t F^{ij} + \partial_k F^{ijk} = P^{<ij>} \\
&\partial_t F^{ill} + \partial_k F^{illk} = P^{ill} \\
&\partial_t F^{iill} + \partial_k F^{iillk} = P^{iill},
\end{aligned} \tag{1}$$

where the independent variables are F, F^i, F^{ij}, F^{ill}, F^{iill}, which are symmetric tensors. See also ref. [2] for further details. The right hand sides of eqs. $(1)_{1,2}$ are zero, such as the trace of that that in eq. $(1)_3$ for the conservation laws of mass, momentum and energy. The entropy principle for these equations, by

using Liu's theorem [3], ensures the existence of parameters called Lagrange multipliers, or mean field, such that

$$\begin{aligned}
& dh = \lambda dF + \lambda_i dF^i + \lambda_{ij} dF^{ij} + \lambda_{ill} dF^{ill} + \lambda_{iill} dF^{iill} \\
& d\phi^k = \lambda dF^k + \lambda_i dF^{ki} + \lambda_{ij} dF^{kij} + \lambda_{ill} dF^{kill} + \lambda_{iill} dF^{kiill} \\
& \sigma = \lambda_{ij} P^{< ij >} + \lambda_{ill} P^{ill} + \lambda_{iill} P^{iill} \geq 0.
\end{aligned} \tag{2}$$

Following the idea exposed in ref. [4], we take the components of the mean field as independent variables and define

$$\begin{aligned}
& h' = \lambda F + \lambda_i F^i + \lambda_{ij} F^{ij} + \lambda_{ill} F^{ill} + \lambda_{iill} F^{iill} - h \\
& \phi'^k = \lambda F^k + \lambda_i F^{ki} + \lambda_{ij} F^{kij} + \lambda_{ill} F^{kill} + \lambda_{iill} F^{kiill} - \phi^k.
\end{aligned} \tag{3}$$

By differentiating eqs. (3) and using eqs. $(2)_{1,2}$ we obtain

$$\begin{aligned}
& dh' = F d\lambda + F^i d\lambda_i + F^{ij} d\lambda_{ij} + F^{ill} d\lambda_{ill} + F^{iill} d\lambda_{iill} \\
& d\phi'^k = F^k d\lambda + F^{ki} d\lambda_i + F^{kij} d\lambda_{ij} + F^{kill} d\lambda_{ill} + F^{kiill} d\lambda_{iill}.
\end{aligned} \tag{4}$$

In the next section a new methodology recently proposed by Pennisi and Ruggeri [5] will be applied (see also [6]) to investigate eqs. (4) together to those expressing the Galilean Relativity principle, showing that they are equivalent to the subsequent conditions (14), (15) and (12). The last one of these will be investigated in section 3 while the other two in section 4. To this end we will need the expansion of h' and ϕ'^k up to whatever order with respect to equilibrium; it will be introduced also in the next section.
In section 5 it will be shown how the 13 moments model can be obtained as a subsystem of the present one.
In section 6 we will see that the results of the kinetic approach are a particular case of those here found with the macroscopic approach.
Finally conclusions will be drawn.

2. The Galilean relativity principle and the entropy principle

We want now to impose the galilean relativity principle. To this end we recall firstly how variables transform with a change of galileanly equivalent frames with relative velocity $\underline{v}$. For the independent variables, from refs. [1], [2], [7] we have

$$\begin{aligned}
F &= m \\
F_i &= m_i + m v_i \\
F_{ij} &= m_{ij} + 2 m_{(i} v_{j)} + m v_i v_j \\
F_{ill} &= m_{ill} + m_{ll} v_i + 2 m_{il} v_l + m_i v^2 + 2 m_l v_l v_i + m v^2 v_i \\
F_{iill} &= m_{iill} + 4 m_{ill} v_i + 2 m_{ll} v^2 + 4 m_{li} v_i v_l + 4 m_l v_l v^2 + m v^4;
\end{aligned} \tag{5}$$

here the $m_{\ldots}$ are the tensors corresponding to $F_{\ldots}$ in the second reference frame.

Moreover we have

$$
\begin{aligned}
F_k &= F v_k + m_k \\
F_{ik} &= F_i v_k + m_{ik} + m_k v_i \\
F_{ijk} &= F_{ij} v_k + m_{ijk} + 2m_{k(i} v_{j)} + m_k v_i v_j \\
F_{illk} &= F_{ill} v_k + m_{illk} + m_{kll} v_i + 2m_{kil} v_l + m_{ki} v^2 + 2m_{kl} v_l v_i + m_k v^2 v_i \\
F_{iillk} &= F_{iill} v_k + m_{iillk} + 4m_{kill} v_i + 2m_{kll} v^2 + 4m_{kli} v_l v_i + 4m_{kl} v_l v^2 + m_k v^4 \\
h &= \hat{h} \\
\phi^k &= \hat{h} v_k + \hat{\phi}^k. \qquad (6)
\end{aligned}
$$

The first two of these, as the trace of the third and fourth ones, are identities, while what remains is the transformation law of the dependent variables. Substituting the relations above into eq. $(2)_1$ and defining

$$
\begin{aligned}
&\hat{\lambda} = \lambda + \lambda_i v_i + \lambda_{ij} v_i v_j + \lambda_{ipp} v^2 v_i + \lambda_{ppqq} v^4 \\
&\hat{\lambda}_i = \lambda_i + 2\lambda_{ij} v_j + 2\lambda_{jpp} v_j v_i + \lambda_{ipp} v^2 + 4\lambda_{ppqq} v^2 v_i \\
&\hat{\lambda}_{ij} = \lambda_{ij} + \lambda_{hpp} v_h \delta_i^j + 2\lambda_{ipp} v_j + 2\lambda_{ppqq} v^2 \delta_i^j + 4\lambda_{ppqq} v_i v_j \\
&\hat{\lambda}_{ill} = \lambda_{ipp} + 4\lambda_{ppqq} v_i \\
&\hat{\lambda}_{ppqq} = \lambda_{ppqq} \qquad (7)
\end{aligned}
$$

we have

$$d\hat{h} = \hat{\lambda} dm + \hat{\lambda}_i dm_i + \hat{\lambda}_{ij} dm_{ij} + \hat{\lambda}_{ill} dm_{ill} + \hat{\lambda}_{iill} dm_{iill}. \qquad (8)$$

For eq. $(6)_6$ we note that eq. (8) is the counterpart of eq. $(2)_1$ in the second frame; this allows us to see that eqs. (7) are the transformation rules for the Lagrange Multipliers.
Similarly, by substituting eqs. (6) in eq. $(2)_2$, we find

$$d\hat{\phi}^k = \hat{\lambda} dm_k + \hat{\lambda}_i dm_{ki} + \hat{\lambda}_{ij} dm_{kij} + \hat{\lambda}_{ill} dm_{kill} + \hat{\lambda}_{iill} dm_{kiill} \qquad (9)$$

which is the counterpart of eq. $(2)_2$ in other frame.
The counterparts of eqs. (3) in the second frame are

$$
\begin{aligned}
\hat{h}' &= m\hat{\lambda} + m_i \hat{\lambda}_i + m_{ij} \hat{\lambda}_{ij} + m_{ill} \hat{\lambda}_{ill} + m_{iill} \hat{\lambda}_{iill} - \hat{h} \\
\hat{\phi}'^k &= m_k \hat{\lambda} + m_{ki} \hat{\lambda}_i + m_{kij} \hat{\lambda}_{ij} + m_{kill} \hat{\lambda}_{ill} + m_{kiill} \hat{\lambda}_{iill} - \hat{\phi}^k; \qquad (10)
\end{aligned}
$$

differentiating them and using eqs. $(6)_{6,7}$, (8) and (9) we obtain respectively

$$
\begin{aligned}
d\hat{h}' &= m d\hat{\lambda} + m_i d\hat{\lambda}_i + m_{ij} d\hat{\lambda}_{ij} + m_{ill} d\hat{\lambda}_{ill} + m_{iill} d\hat{\lambda}_{iill}, \\
d\hat{\phi}'^k &= m_k d\hat{\lambda} + m_{ki} d\hat{\lambda}_i + m_{kij} d\hat{\lambda}_{ij} + m_{kill} d\hat{\lambda}_{ill} + m_{kiill} d\hat{\lambda}_{iill}.
\end{aligned}
$$

Taking their derivatives with respect to the various components of the main field we have

$$
\begin{aligned}
&m = \frac{\partial \hat{h}'}{\partial \hat{\lambda}}, \quad m_i = \frac{\partial \hat{h}'}{\partial \hat{\lambda}_i}, \quad m_{ij} = \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ij}}, \quad m_{ill} = \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ill}}, \quad m_{iill} = \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{iill}}, \\
&m_k = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}}, \quad m_{ki} = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_i}, \quad m_{kij} = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{ij}}, \quad m_{kill} = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{ill}}, \quad m_{kiill} = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{iill}}.
\end{aligned}
\qquad (11)
$$

Comparing the correspondent terms in the two rows of eq. (11) we obtain the following compatibility conditions:

$$\frac{\partial \hat{h}'}{\partial \hat{\lambda}_k} = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}}, \qquad \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ki}} = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_i}, \qquad \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ill}} = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{ij}} \delta_i^j,$$
$$\frac{\hat{\phi}'^{k]}}{\partial \hat{\lambda}_{i[j}} = 0, \qquad \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{kkll}} = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{ill}} \delta_i^k, \qquad \frac{\partial \hat{\phi}'^{k]}}{\partial \hat{\lambda}_{ll[i}} = 0. \tag{12}$$

By substituting h, $\hat{h}$, ϕ^k and $\hat{\phi}^k$ from eqs. $(3)_1$, $(10)_1$, $(3)_2$, $(10)_2$ into eqs. $(6)_{6,7}$, these become

$$h' = \hat{h}', \qquad \phi'^k = \hat{\phi}'^k + \hat{h}' v^k, \tag{13}$$

where $(6)_{1-5}$ and (7) have been used.

Now, from eqs. (13) we see that h' and ϕ'^k are composite functions of $\hat{h}'$ and $\hat{\phi}'^k$ and of eqs. (7); but h' and ϕ'^k depend only on λ, λ_i, λ_{ij}, λ_{ill}, λ_{iill} and not on v_h. In other words, the derivative of h' and ϕ'^k with respect to v_h, through the above mentioned composite functions, must be zero, i.e.

$$\frac{\partial h'}{\partial v_h} = 0 = m\hat{\lambda}_h + 2m_i\hat{\lambda}_{ih} + \hat{\lambda}_{ipp}\left(m_{ll}\delta_i^h + 2m_{ih}\right) + 4m_{hll}\hat{\lambda}_{ppqq} \tag{14}$$

$$\frac{\partial \phi'^k}{\partial v_h} = 0 = m_k\hat{\lambda}_h + 2m_{ki}\hat{\lambda}_{ih} + \hat{\lambda}_{ipp}\left(m_{kll}\delta_i^h + 2m_{kih}\right) + 4m_{khll}\hat{\lambda}_{ppqq} + \delta_h^k h' \tag{15}$$

where (11) and (7) have been used.

The entropy principle and that of material objectivity reduce in imposing eqs. (14), (15) and (12). We want to impose these conditions up to whatever order with respect to thermodynamical equilibrium. This is defined, see [8], as the state where all the components of the main field, except $\hat{\lambda}$ and $\hat{\lambda}_{ij} = \frac{1}{3}\hat{\lambda}_{ll}\delta_{ij}$, amounts to zero. To avoid an excessive quantity of indexes, we will do later the expansion with respect to $\hat{\lambda}_{ppqq}$. The expansion of the tensor $\hat{\phi}'^i$ with respect to the other variables is

$$\hat{\phi}'^i = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty} \frac{1}{p!q!r!} \phi_{p,q,r}^{ii_1\cdots i_p j_1\cdots j_q k_1 h_1 \cdots k_r h_r} \hat{\lambda}_{i_1}\cdots\hat{\lambda}_{i_p}\hat{\lambda}_{j_1 ll}\cdots\hat{\lambda}_{j_q ll} \cdot$$
$$\left(\hat{\lambda}_{k_1h_1} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_1h_1}\right)\cdots\left(\hat{\lambda}_{k_rh_r} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_rh_r}\right) \tag{16}$$

with
$$\phi_{p,q,r}^{ii_1\cdots i_p j_1\cdots j_q k_1 h_1 \cdots k_r h_r}(\hat{\lambda}, \hat{\lambda}_{ll}, \hat{\lambda}_{ppqq}) =$$
$$= \left(\frac{\partial^{p+q+r}\hat{\phi}'^i}{\partial\hat{\lambda}_{i_1}\cdots\partial\hat{\lambda}_{i_p}\partial\hat{\lambda}_{j_1ll}\cdots\partial\hat{\lambda}_{j_qll}\partial\hat{\lambda}_{k_1h_1}\cdots\partial\hat{\lambda}_{h_rk_r}}\right)_{eq}. \tag{17}$$

Now, from the compatibility conditions $(12)_2$, $(12)_6$ and $(12)_4$ we see that we can exchange the index i respectively with each other index taken from $i_1\cdots i_p$, $j_1,\cdots j_q$ and $h_1\cdots h_r$ or $k_1\cdots k_r$, so $\phi_{p,q,r}^{ii_1\cdots i_p j_1\cdots j_q k_1 h_1 \cdots k_r h_r}$ is a symmetric tensor with respect to any couple of indexes. Moreover $\phi_{p,q,r}^{ii_1\cdots i_p j_1\cdots j_q k_1 h_1 \cdots k_r h_r}$ depends only on scalars, so that

$$\begin{cases} \phi_{p,q,r}^{ii_1\cdots i_p j_1\cdots j_q k_1 h_1 \cdots k_r h_r} = 0 & \text{if p+q+2r+1 is odd} \\ \phi_{p,q,r}^{ii_1\cdots i_p j_1\cdots j_q k_1 h_1 \cdots k_r h_r} = \phi_{p,q,r}(\hat{\lambda}, \hat{\lambda}_{ll}, \hat{\lambda}_{ppqq})\delta^{ii_1}\cdots\delta^{k_rh_r} & \text{if p+q+2r+1 is even,} \end{cases} \tag{18}$$

so that $\phi_{p,q,r}^{ii_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}$ is known except for a scalar function.
Similarly, for the tensor $\hat{h}'$ we have

$$\hat{h}' = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\frac{1}{p!q!r!}h_{p,q,r}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}\hat{\lambda}_{i_1}\cdots\hat{\lambda}_{i_p}\hat{\lambda}_{j_1 ll}\cdots\hat{\lambda}_{j_q ll}\cdot \left(\hat{\lambda}_{k_1 h_1}-\frac{1}{3}\hat{\lambda}_{ll}\delta_{k_1 h_1}\right)\cdots\left(\hat{\lambda}_{k_r h_r}-\frac{1}{3}\hat{\lambda}_{ll}\delta_{k_r h_r}\right) \tag{19}$$

with
$$h_{p,q,r}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}(\hat{\lambda},\hat{\lambda}_{ll},\hat{\lambda}_{ppqq}) = = \left(\frac{\partial^{p+q+r}\hat{h}'}{\partial\hat{\lambda}_{i_1}\cdots\partial\hat{\lambda}_{i_p}\partial\hat{\lambda}_{j_1 ll}\cdots\partial\hat{\lambda}_{j_q ll}\partial\hat{\lambda}_{k_1 h_1}\cdots\partial\hat{\lambda}_{h_r k_r}}\right)_{eq}. \tag{20}$$

Taking the derivatives with respect to $\hat{\lambda}_{jll}$ of the compatibility conditions $(12)_1$ and $(12)_2$ and using $(12)_6$ we see that we can exchange every index taken from $j_1,\cdots j_q$ with each other. Similarly, taking the derivative of eq. $(12)_1$ with respect to $\hat{\lambda}_{rs}$ and using eq. $(12)_4$ we see that we can exchange every index taken from $i_1,\cdots,i_p$ with each other. Consequently, $h_{p,q,r}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}$ is a symmetric tensor with respect to any couple of indexes; moreover it depends only on scalars, so that

$$\begin{cases} h_{p,q,r}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r} = 0 & \text{if p+q+2r is odd} \\ h_{p,q,r}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r} = h_{p,q,r}(\hat{\lambda},\hat{\lambda}_{ll},\hat{\lambda}_{ppqq})\delta^{i_1 i_2}\cdots\delta^{k_r h_r} & \text{if p+q+2r is even.} \end{cases} \tag{21}$$

In other words, also $h_{p,q,r}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}$ is known except for a scalar function.
We want to avoid to use eqs. (17) and (20) in the sequel. To this end we note that we can consider

$$\frac{\partial^{p+q+r}\hat{h}'}{\partial\hat{\lambda}_{i_1}\cdots\partial\hat{\lambda}_{i_p}\partial\hat{\lambda}_{j_1 ll}\cdots\partial\hat{\lambda}_{j_q ll}\partial\hat{\lambda}_{k_1 h_1}\cdots\partial\hat{\lambda}_{h_r k_r}}$$

and
$$\frac{\partial^{p+q+r}\hat{\phi}'^k}{\partial\hat{\lambda}_{i_1}\cdots\partial\hat{\lambda}_{i_p}\partial\hat{\lambda}_{j_1 ll}\cdots\partial\hat{\lambda}_{j_q ll}\partial\hat{\lambda}_{k_1 h_1}\cdots\partial\hat{\lambda}_{h_r k_r}}$$

depending on $\hat{\lambda}_{ab}$ as composite functions through $\hat{\lambda}_{<ab>} = \left(\delta_a^i\delta_b^j - \frac{1}{3}\delta^{ij}\delta_{ab}\right)\hat{\lambda}_{ij}$ and $\hat{\lambda}_{ll}$. With this in mind let us take their derivatives with respect to $\hat{\lambda}_{ab}$, after that contract them with δ_{ab} and calculate the result at equilibrium; we find

$$h_{p,q,r+1}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r ab}\delta_{ab} = 3\frac{\partial}{\partial\hat{\lambda}_{ll}}h_{p,q,r}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r} \tag{22}$$

and
$$\phi_{p,q,r+1}^{ii_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r ab}\delta_{ab} = 3\frac{\partial}{\partial\hat{\lambda}_{ll}}\phi_{p,q,r}^{ii_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}. \tag{23}$$

An interesting consequence of eq. (22) can be observed as follows.
Let us take the derivative of $\hat{h}'$ with respect to $\hat{\lambda}_{ij}$ taking into account that $\hat{\lambda}_{ij} = \frac{1}{3}\hat{\lambda}_{ll}\delta_{ij} + \hat{\lambda}_{<ij>}$:

$$\frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ij}} = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty} \frac{1}{p!q!r!} \frac{\partial h_{p,q,r}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r}}{\partial \hat{\lambda}_{ll}} \hat{\lambda}_{i_1}\cdots\hat{\lambda}_{i_p}\hat{\lambda}_{j_1 ll}\cdots\hat{\lambda}_{j_q ll}\cdot$$
$$\left(\hat{\lambda}_{k_1h_1} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_1h_1}\right)\cdots\left(\hat{\lambda}_{k_rh_r} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_rh_r}\right)\delta_{ij} +$$
$$+\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=1}^{\infty} \frac{r}{p!q!r!} h_{p,q,r}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r} \hat{\lambda}_{i_1}\cdots\hat{\lambda}_{i_p}\hat{\lambda}_{j_1 ll}\cdots\hat{\lambda}_{j_q ll}\cdot$$
$$\left(\hat{\lambda}_{k_1h_1} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_1h_1}\right)\cdots\left(\hat{\lambda}_{k_{r-1}h_{r-1}} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_{r-1}h_{r-1}}\right)\left(\delta^i_{h_r}\delta^j_{k_r} - \frac{1}{3}\delta_{h_rk_r}\delta^{ij}\right),$$

which, by using eq. (22), becomes

$$\begin{aligned}
\frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ij}} &= \Bigg[\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty} \frac{1}{p!q!r!}\frac{1}{3} h_{p,q,r+1}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r ab}\delta_{ab}\hat{\lambda}_{i_1}\cdots\hat{\lambda}_{i_p}\cdot \\
& \hat{\lambda}_{j_1 ll}\cdots\hat{\lambda}_{j_q ll}\left(\hat{\lambda}_{k_1h_1} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_1h_1}\right)\cdots\left(\hat{\lambda}_{k_rh_r} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_rh_r}\right)\delta^{ij} + \\
- & \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=1}^{\infty} \frac{r}{p!q!r!}\frac{1}{3} h_{p,q,r}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r} \hat{\lambda}_{i_1}\cdots\hat{\lambda}_{i_p}\hat{\lambda}_{j_1 ll}\cdots\hat{\lambda}_{j_q ll}\cdot \\
& \left(\hat{\lambda}_{k_1h_1} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_1h_1}\right)\cdots\left(\hat{\lambda}_{k_{r-1}h_{r-1}} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_{r-1}h_{r-1}}\right)\delta_{h_rk_r}\Bigg]\delta^{ij} + \\
+ & \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=1}^{\infty} \frac{r}{p!q!r!} h_{p,q,r}^{i_1\cdots i_p j_1\cdots j_q k_1 h_1\cdots k_r h_r} \hat{\lambda}_{i_1}\cdots\hat{\lambda}_{i_p}\hat{\lambda}_{j_1 ll}\cdots\hat{\lambda}_{j_q ll}\cdot \\
& \left(\hat{\lambda}_{k_1h_1} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_1h_1}\right)\cdots\left(\hat{\lambda}_{k_{r-1}h_{r-1}} - \frac{1}{3}\hat{\lambda}_{ll}\delta_{k_{r-1}h_{r-1}}\right)\delta^i_{h_r}\delta^j_{k_r};
\end{aligned}$$

We note that the term in square brackets amounts to zero as can be easily proved by substituting r=R+1 in the second sum. What remains can be written as

$$\frac{\partial \hat{h}'}{\partial \hat{\lambda}_{ij}} = \frac{\partial \hat{h}'}{\partial \hat{\lambda}_{<ij>}},$$

where the derivative in the right hand side has been taken without considering that the components of $\hat{\lambda}_{<ij>}$ aren't independent because restricted by $\hat{\lambda}_{<ij>}\delta^{ij} = 0$. Proceeding similarly with $\hat{\phi}'^k$ and using eq. (23) we find that

$$\frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{ij}} = \frac{\partial \hat{\phi}'^k}{\partial \hat{\lambda}_{<ij>}}.$$

After that, we see that eq. (17) and (20) become consequences of eqs. (16) and (19) so that they can be forgotten. But, instead of them, we have to impose eqs. (22) and (23).
Expliciting eq. (22) by means of eq. (21) we have

$$h_{p,q,r+1} = 3\frac{p+q+2r+1}{p+q+2r+3}\frac{\partial h_{p,q,r}}{\partial \hat{\lambda}_{ll}}, \tag{24}$$

from which

$$h_{p,q,r} = 3^r \frac{p+q+1}{p+q+2r+1} \frac{\partial^r h_{p,q,0}}{\partial \hat{\lambda}^r_{ll}}, \tag{25}$$

as it can be seen by using the iterative procedure.
Similarly, expliciting eq. (23) by means of eq. (18), we have

$$\phi_{p,q,r+1} = 3 \frac{p+q+2r+2}{p+q+2r+4} \frac{\partial \phi_{p,q,r}}{\partial \hat{\lambda}_{ll}}, \tag{26}$$

from which

$$\phi_{p,q,r} = 3^r \frac{p+q+2}{p+q+2r+2} \frac{\partial^r \phi_{p,q,0}}{\partial \hat{\lambda}^r_{ll}}, \tag{27}$$

that can be proved using the iterative procedure.
If we introduce the quantities

$$\begin{cases} k_{p,q} = h_{p,q,0} & \text{if p+q is even} \\ k_{p,q} = \phi_{p,q,0} & \text{if p+q is odd,} \end{cases} \tag{28}$$

we note that $\hat{h}'$ and $\hat{\phi}'^k$ are known if we know all the terms of the infinity matrix $k_{p,q}$; so our aim is to find $k_{p,q}$. We have also to impose the compatibility conditions (12) and the conditions (14) and (15) expressing the Galilean relativity principle. Let us begin by investigating the conditions (12).

3. Exploitation of the conditions (12)

Now let's impose conditions (12) on our tensors. We notice that equations $(12)_{4,6}$ are already satisfied because the tensors $\phi^{ii_1\cdots i_p j_1\cdots j_q h_1 k_1\cdots h_r k_r}_{p,q,r}$ are symmetric, so there remains to impose eqs. $(12)_{1,2,3,5}$.

- Eq. $(12)_1$, by using (16), (18), (20) and (21), becomes

$$h_{p+1,q,r} = \frac{\partial \phi_{p,q,r}}{\partial \hat{\lambda}} \tag{29}$$

which, for r=0 reads

$$h_{p+1,q,0} = \frac{\partial \phi_{p,q,0}}{\partial \hat{\lambda}} \tag{30}$$

and, for the other values of r is consequence of (25), (27), (30). This last one, by using (28), can be written also as

$$k_{p+1,q} = \frac{\partial k_{p,q}}{\partial \hat{\lambda}} \quad \text{with p+q+1 even.} \tag{31}$$

In other words, the elements with p+q+1 even of the matrix $k_{p+1,q}$ can be expressed in terms of that of the same column but previous row.

- Let us impose now eq. $(12)_2$, using eqs. (16), (18), (20) and (21); we obtain

$$h_{p,q,r+1} = \phi_{p+1,q,r} \tag{32}$$

which, by using eqs. (25) and (27) is equivalent to

$$\phi_{p+1,q,0} = 3 \frac{p+q+1}{p+q+3} \frac{\partial h_{p,q,0}}{\partial \hat{\lambda}_{ll}} \tag{33}$$

and this, by using (28), becomes

$$k_{p+1,q} = 3\frac{p+q+1}{p+q+3}\frac{\partial k_{p,q}}{\partial \hat{\lambda}_{ll}} \quad \text{with p+q even.} \tag{34}$$

Using (31) or (34) we can express all the elements of the matrix $k_{p,q}$ in terms of those in the same column and previous row. Iterating this procedure each element can be expressed in terms of the elements in the first row of the matrix. In fact joining eqs. (31) and (34) we obtain

$$\begin{cases} k_{p,q} = 3^{\frac{p}{2}}\,\frac{q+1}{p+q+1}\,\frac{\partial^p}{\partial \hat{\lambda}_{ll}{}^{\frac{p}{2}}\,\partial \hat{\lambda}^{\frac{p}{2}}}k_{0,q} & \text{with p and q even,} \\ k_{p,q} = 3^{\frac{p-1}{2}}\,\frac{q+2}{p+q+1}\,\frac{\partial^p}{\partial \hat{\lambda}_{ll}{}^{\frac{p-1}{2}}\,\partial \hat{\lambda}^{\frac{p+1}{2}}}k_{0,q} & \text{with p and q odd,} \\ k_{p,q} = 3^{\frac{p}{2}}\,\frac{q+2}{p+q+2}\,\frac{\partial^p}{\partial \hat{\lambda}_{ll}{}^{\frac{p}{2}}\,\partial \hat{\lambda}^{\frac{p}{2}}}k_{0,q} & \text{with p even and q odd,} \\ k_{p,q} = 3^{\frac{p+1}{2}}\,\frac{q+1}{p+q+2}\,\frac{\partial^p}{\partial \hat{\lambda}_{ll}{}^{\frac{p+1}{2}}\,\partial \hat{\lambda}^{\frac{p-1}{2}}}k_{0,q} & \text{with p odd and q even.} \end{cases} \tag{35}$$

- Finally, let us consider eqs. $(12)_{3,5}$. Using eqs. (16), (18), (20) and (21) they become respectively

$$h_{p,q+1,r} = \frac{p+q+2r+4}{p+q+2r+2}\phi_{p,q,r+1} \tag{36}$$

and

$$\frac{\partial h_{p,q,r}}{\partial \hat{\lambda}_{kkll}} = \frac{p+q+2r+3}{p+q+2r+1}\phi_{p,q+1,r}. \tag{37}$$

By using eqs. (25), (27) and finally (28) the above equations transform respectively into

$$k_{p,q+1} = 3\frac{\partial k_{p,q}}{\partial \hat{\lambda}_{ll}} \quad \text{with p+q+1 even} \tag{38}$$

and

$$k_{p,q+1} = \frac{p+q+1}{p+q+3}\frac{\partial k_{p,q}}{\partial \hat{\lambda}_{aabb}} \quad \text{with p+q+1 odd.} \tag{39}$$

In other words with eqs. (38) and (39) each element of the matrix $k_{p,q}$ can be written in terms of the element in the same row and previous column. But we already know, by eqs. (35), each row of the matrix $k_{p,q}$ in terms of the first one; so we have to investigate the compatibility of these two results. By substituting eqs. (35) into eqs. (38) and (39) we obtain a series of equations for the first row of the matrix $k_{p,q}$, i.e.,

$$\begin{cases} k_{0,q+1} = 3\frac{\partial}{\partial \hat{\lambda}_{ll}}k_{0,q} & \text{q odd,} \\ 9\frac{q+1}{q+3}\,\frac{\partial^2}{\partial \hat{\lambda}_{ll}^2}k_{0,q} = \frac{\partial}{\partial \hat{\lambda}}k_{0,q+1} & \text{q even,} \\ \frac{q+1}{q+3}\,\frac{\partial}{\partial \hat{\lambda}_{aabb}}k_{0,q} = k_{0,q+1} & \text{q even,} \\ \frac{\partial}{\partial \hat{\lambda}_{aabb}}\,\frac{\partial}{\partial \hat{\lambda}}k_{0,q} = 3\frac{\partial}{\partial \hat{\lambda}_{ll}}k_{0,q+1} & \text{q odd,} \end{cases} \tag{40}$$

and other equations which are consequences of these last ones. Now eqs. $(40)_1$ and $(40)_3$ give each element $k_{0,q}$ in terms of $k_{0,0}$, i.e.,

$$k_{0,q} = 3^{\frac{q}{2}}\frac{1}{q+1}\frac{\partial^q}{\partial \hat{\lambda}_{ll}^{\frac{q}{2}}\partial \hat{\lambda}_{aabb}^{\frac{q}{2}}}k_{0,0} \quad \text{with q even} \tag{41}$$

$$k_{0,q} = 3^{\frac{q-1}{2}} \frac{1}{q+2} \frac{\partial^q}{\partial \hat{\lambda}_{ll}^{\frac{q-1}{2}} \partial \hat{\lambda}_{aabb}^{\frac{q+1}{2}}} k_{0,0} \quad \text{with q odd.} \tag{42}$$

Through these two equations is possible to express a generic element in the first row in terms of the first element in the same first row of the matrix.
Eq. $(40)_{2,4}$ remain to be imposed. The first one of these with q=0 and by use of (41) reads

$$9 \frac{\partial^2}{\partial \hat{\lambda}_{ll}^2} k_{0,0} = \frac{\partial}{\partial \hat{\lambda}} \frac{\partial}{\partial \hat{\lambda}_{aabb}} k_{0,0}, \tag{43}$$

which is a condition on $k_{0,0}$. After that eq. $(40)_2$ for the other values of q is a consequence of eq. (43).
At last, eq. $(40)_4$ with use of (41) and (42) becomes equivalent to its value for q=1, i.e.,

$$9 \frac{\partial^3}{\partial \hat{\lambda}_{ll}^2 \partial \hat{\lambda}_{aabb}} k_{0,0} = \frac{\partial^3}{\partial \hat{\lambda} \partial \hat{\lambda}_{aabb}^2} k_{0,0},$$

which is eq. (43) differentiated with respect to $\hat{\lambda}_{aabb}$; so it is sufficient to impose eq. (43).
We can now substitute eqs. (41) and (42) into eqs. (35) which now become

$$\begin{cases} k_{p,q} = 3^{\frac{p+q}{2}} \frac{1}{p+q+1} \frac{\partial^{p+q}}{\partial \hat{\lambda}_{ll}^{\frac{p+q}{2}} \partial \hat{\lambda}^{\frac{p}{2}} \partial \hat{\lambda}_{aabb}^{\frac{q}{2}}} k_{0,0} & \text{with p and q even,} \\ k_{p,q} = 3^{\frac{p+q-2}{2}} \frac{1}{p+q+1} \frac{\partial^{p+q}}{\partial \hat{\lambda}_{ll}^{\frac{p+q-2}{2}} \partial \hat{\lambda}^{\frac{p+1}{2}} \partial \hat{\lambda}_{aabb}^{\frac{q+1}{2}}} k_{0,0} & \text{with p and q odd,} \\ k_{p,q} = 3^{\frac{p+q-1}{2}} \frac{1}{p+q+2} \frac{\partial^{p+q}}{\partial \hat{\lambda}_{ll}^{\frac{p+q-1}{2}} \partial \hat{\lambda}^{\frac{p}{2}} \partial \hat{\lambda}_{aabb}^{\frac{q+1}{2}}} k_{0,0} & \text{with p even and q odd,} \\ k_{p,q} = 3^{\frac{p+q+1}{2}} \frac{1}{p+q+2} \frac{\partial^{p+q}}{\partial \hat{\lambda}_{ll}^{\frac{p+q+1}{2}} \partial \hat{\lambda}^{\frac{p-1}{2}} \partial \hat{\lambda}_{aabb}^{\frac{q}{2}}} k_{0,0} & \text{with p odd and q even.} \end{cases} \tag{44}$$

In this way all the elements of the matrix $k_{p,q}$ are determined in terms of $k_{0,0}$ which is restricted, until now, only by eq. (43). Another restriction will be found in the next section.

4. Exploitation of the conditions (14) and (15)

There remains now to impose eqs. (14) and (15), but we can see that (14) is a consequence of (15) and (12). In fact

- the derivative of (14) with respect to $\hat{\lambda}_k$ is equal to the derivative of (15) with respect to λ, thanks to (11), $(12)_1$,
- the derivative of (14) with respect to $\hat{\lambda}_{kb}$ is exactly the derivative of (15) with respect to λ_b, thanks to (11), $(12)_{2,1}$,
- the derivative of (14) with respect to $\hat{\lambda}_{kll}$ is exactly the derivative of (15) with respect to λ_{ab}, contracted after derivation by δ_{ab}, thanks to (11), $(12)_{3,2}$,
- the derivative of (14) with respect to $\hat{\lambda}_{kkll}$ is exactly the derivative of (15) with respect to λ_{ill}, contracted after derivation by δ_{ki}, thanks to (11), $(12)_5$.

Consequently, eq. (14) needs to be imposed only for $\hat{\lambda}_k = 0$, $\hat{\lambda}_{ab} = 0$, $\hat{\lambda}_{kll} = 0$ and $\hat{\lambda}_{kkll} = 0$, and in this case it is an identity. So it remains to impose only

eq. (15). To this end it is useful to use the identity

$$\frac{\partial^r}{\partial\hat\lambda_{k_1h_1}\cdots\partial\hat\lambda_{k_rh_r}}\left(\hat\lambda_{ij}\frac{\partial\phi'^k}{\partial\hat\lambda_i}\right) = \hat\lambda_{ij}\frac{\partial^{r+1}\phi'^k}{\partial\hat\lambda_i\partial\hat\lambda_{k_1h_1}\cdots\partial\hat\lambda_{k_rh_r}} + r\delta_{j(k_1}\frac{\partial^r\phi'^k}{\partial\hat\lambda_{h_1}\partial\hat\lambda_{k_2h_2}\cdots\partial\hat\lambda_{k_rh_r)}}$$

whose proof can be found in the Appendix of ref. [5] and holds also if, in our case, $\hat\phi'^k$ depends on the further independent variable $\hat\lambda_{aabb}$.
Let us take now the derivative of eq. (15) with respect to $\hat\lambda_{i_1}\cdots\hat\lambda_{i_p}$, $\hat\lambda_{j_1ll}\cdots\hat\lambda_{j_qll}$, $\hat\lambda_{k_1h_1}\cdots\hat\lambda_{k_rh_r}$. If we calculate it at equilibrium and we use eqs. (17) and (20) we obtain

$$\begin{aligned} & p\delta_{h(i_1}\frac{\partial}{\partial\hat\lambda}\phi^{i_2\cdots i_p)kj_1\cdots j_qh_1k_1\cdots h_rk_r}_{p-1,q,r} + \frac{2}{3}\hat\lambda_{ll}\phi^{khi_1\cdots i_pj_1\cdots j_qh_1k_1\cdots h_rk_r}_{p+1,q,r} + \\ + \; & 2r\delta_{h(k_1}\phi^{h_1k_2\cdots h_rk_r)ki_1\cdots i_pj_1\cdots j_q}_{p+1,q,r-1} + 2q\phi^{khi_1\cdots i_pj_1\cdots j_qh_1k_1\cdots h_rk_r}_{p,q-1,r+1} + \\ + \; & q\delta_{h(j_1}\phi^{j_2\cdots j_q)ki_1\cdots i_ph_1k_1\cdots h_rk_rab}_{p,q-1,r+1}\delta_{ab} + 4\hat\lambda_{aabb}\phi^{ki_1\cdots i_pj_1\cdots j_qhh_1k_1\cdots h_rk_r}_{p,q+1,r} + \\ + \; & h^{i_1\cdots i_pj_1\cdots j_qh_1k_1\cdots h_rk_r}_{p,q,r}\delta^{hk} = 0. \end{aligned} \tag{45}$$

To evaluate this condition it will be useful to do the following considerations:

1) Let $\psi^{\cdots}$ be a symmetric tensor; it is easy to prove that

$$\begin{aligned} \delta^{h(i_1}\psi^{i_2\cdots i_pj_1\cdots j_qe_1\cdots e_sk)} = \; & \frac{p}{p+q+s+1}\delta^{h(i_1}\psi^{i_2\cdots i_p)j_1\cdots j_qe_1\cdots e_sk} + \\ + \; & \frac{q}{p+q+s+1}\delta^{h(j_1}\psi^{j_2\cdots j_q)i_1\cdots i_pe_1\cdots e_sk} + \\ + \; & \frac{s}{p+q+s+1}\delta^{h(e_1}\psi^{e_2\cdots e_s)i_1\cdots i_pj_1\cdots j_qk} + \\ + \; & \frac{1}{p+q+s+1}\delta^{hk}\psi^{i_1\cdots i_pj_1\cdots j_qe_1\cdots e_s}. \end{aligned}$$

2) Moreover we have

$$\phi^{j_2\cdots j_qki_1\cdots i_ph_1k_1\cdots h_rk_rab}_{p,q-1,r+1}\delta_{ab} = \phi_{p,q-1,r+1}\frac{q+p+2r+3}{q+p+2r+1}\delta^{(j_2\cdots j_qki_1\cdots i_ph_1k_1\cdots h_rk_r)}.$$

3) Finally, we can express everything in terms of the scalar $h_{p,q,r}$ using the following relations:

$$\begin{cases} \frac{\partial}{\partial\hat\lambda}\phi_{p-1,q,r} = h_{p,q,r} & \text{from eq. (29)}, \\ \phi_{p+1,q,r-1} = h_{p,q,r}, \quad \phi_{p+1,q,r} = h_{p,q,r+1} & \text{from eq. (32)}, \\ \phi_{p,q-1,r+1} = \frac{p+q+2r+1}{p+q+2r+3}h_{p,q,r} & \text{from eq. (36)}, \\ \phi_{p,q+1,r} = \frac{p+q+2r+1}{p+q+2r+3}\frac{\partial}{\partial\hat\lambda_{aabb}}h_{p,q,r} & \text{from eq. (37)}. \end{cases}$$

All these results allow to rewrite eq. (45) as

$$\begin{aligned} 0 = h_{p,q,r}(p+q+2r+1)\delta^{h(i_1}\delta^{i_2\cdots h_rk_rk)} + \frac{2}{3}\hat\lambda_{ll}\delta^{khi_1\cdots h_rk_r}h_{p,q,r+1} + \\ +2q\frac{p+q+2r+1}{p+q+2r+3}\delta^{khi_1\cdots h_rk_r}h_{p,q,r} + 4\hat\lambda_{aabb}\frac{p+q+2r+1}{p+q+2r+3}\frac{\partial h_{p,q,r}}{\partial\hat\lambda_{aabb}}\delta^{hki_1\cdots h_rk_r}, \end{aligned}$$

where the notation $\delta^{e_1 e_2 \cdots e_{2s}} = \delta^{(e_1 e_2} \cdots \delta^{e_{2s-1} e_{2s})}$ has been used; the result is equivalent to

$$\begin{aligned} 0 &= (p+q+2r+1) h_{p,q,r} + \frac{2}{3} \hat{\lambda}_{ll} h_{p,q,r+1} + \\ &+ \frac{p+q+2r+1}{p+q+2r+3} \left(2q h_{p,q,r} + 4 \hat{\lambda}_{aabb} \frac{\partial h_{p,q,r}}{\partial \hat{\lambda}_{aabb}} \right). \end{aligned}$$

This equation, by using eqs. (25) and (28), becomes

$$0 = (p+3q+2r+3) \frac{\partial^r}{\partial \hat{\lambda}_{ll}^r} k_{p,q} + 2 \hat{\lambda}_{ll} \frac{\partial^{r+1}}{\partial \hat{\lambda}_{ll}^{r+1}} k_{p,q} + 4 \hat{\lambda}_{aabb} \frac{\partial^r}{\partial \hat{\lambda}_{ll}^r} \frac{\partial k_{p,q}}{\partial \hat{\lambda}_{aabb}}$$

with p+q even. We note that if this relation holds until a fixed r taking its derivative with respect to $\hat{\lambda}_{ll}$ we obtain that it holds also with r+1 replacing r. Therefore, it suffices to impose this relation for the lower value of r, i.e for r=0. In this case it becomes

$$0 = (p+3q+3) k_{p,q} + 2 \hat{\lambda}_{ll} \frac{\partial}{\partial \hat{\lambda}_{ll}} k_{p,q} + 4 \hat{\lambda}_{aabb} \frac{\partial k_{p,q}}{\partial \hat{\lambda}_{aabb}}, \tag{46}$$

with p+q even.

Let us firstly analyze the case with p and q even. Putting eq. $(35)_1$ into (46) we have

$$0 = (p+3q+3) \frac{\partial^p}{\partial \hat{\lambda}_{ll}^{\frac{p}{2}} \partial \hat{\lambda}^{\frac{p}{2}}} k_{0,q} + 2 \hat{\lambda}_{ll} \frac{\partial^{p+1}}{\partial \hat{\lambda}_{ll}^{\frac{p}{2}+1} \partial \hat{\lambda}^{\frac{p}{2}}} k_{0,q} + 4 \hat{\lambda}_{aabb} \frac{\partial^p}{\partial \hat{\lambda}_{ll}^{\frac{p}{2}} \partial \hat{\lambda}^{\frac{p}{2}}} \frac{\partial k_{0,q}}{\partial \hat{\lambda}_{aabb}}.$$

We note that if this relation holds until a fixed p taking its derivative with respect to $\hat{\lambda}_{ll}$ and then with respect to $\hat{\lambda}$, we obtain that it holds also with p+2 replacing p (p must be even). Therefore, it suffices to impose this relation for the lower even value of p, i.e for p=0.

In this case it becomes

$$0 = (3q+3) k_{0,q} + 2 \hat{\lambda}_{ll} \frac{\partial}{\partial \hat{\lambda}_{ll}} k_{0,q} + 4 \hat{\lambda}_{aabb} \frac{\partial k_{0,q}}{\partial \hat{\lambda}_{aabb}}, \tag{47}$$

that is (46) calculated in p=0.

By using eq. (41) we see that eq. (47) becomes

$$0 = (3q+3) \frac{\partial^q}{\partial \hat{\lambda}_{ll}^{\frac{q}{2}} \partial \hat{\lambda}_{aabb}^{\frac{q}{2}}} k_{0,0} + 2 \hat{\lambda}_{ll} \frac{\partial^{q+1}}{\partial \hat{\lambda}_{ll}^{\frac{q}{2}+1} \partial \hat{\lambda}_{aabb}^{\frac{q}{2}}} k_{0,0} + 4 \hat{\lambda}_{aabb} \frac{\partial^q}{\partial \hat{\lambda}_{ll}^{\frac{q}{2}} \partial \hat{\lambda}_{aabb}^{\frac{q}{2}}} \frac{\partial k_{0,0}}{\partial \hat{\lambda}_{ccgg}}.$$

We note that if this relation holds until a fixed q taking its derivative with respect to $\hat{\lambda}_{ll}$ and then with respect to $\hat{\lambda}_{aabb}$, we obtain that it holds also with q+2 replacing q (q must be even). Therefore, it suffices to impose this relation for the lower even order of q, i.e for q=0. In this case it becomes

$$0 = 3 k_{0,0} + 2 \hat{\lambda}_{ll} \frac{\partial}{\partial \hat{\lambda}_{ll}} k_{0,0} + 4 \hat{\lambda}_{aabb} \frac{\partial k_{0,0}}{\partial \hat{\lambda}_{aabb}}, \tag{48}$$

that is (46) calculated in p=0, q=0.

There remains the case with p and q odd. We will see that it will give only

identities. In fact, putting eq. $(35)_2$ into (46), this becomes

$$\begin{aligned} 0 &= (p+3q+3)\frac{\partial^p}{\partial\hat\lambda_{ll}^{\frac{p-1}{2}}\partial\hat\lambda^{\frac{p+1}{2}}}k_{0,q}+ \\ &+ 2\hat\lambda_{ll}\frac{\partial^{p+1}}{\partial\hat\lambda_{ll}^{\frac{p-1}{2}+1}\partial\hat\lambda^{\frac{p+1}{2}}}k_{0,q}+4\hat\lambda_{aabb}\frac{\partial^p}{\partial\hat\lambda_{ll}^{\frac{p-1}{2}}\partial\hat\lambda^{\frac{p+1}{2}}}\frac{\partial k_{0,q}}{\partial\hat\lambda_{aabb}}. \end{aligned}$$

We note that if this relation holds until a fixed p taking its derivative with respect to $\hat\lambda_{ll}$ and then with respect to $\hat\lambda$, we obtain that it holds also with p+2 replacing p (p must be odd). Therefore, it suffices to impose this relation for the lower odd value of p, i.e p=1. In this case it becomes

$$0=(3q+4)\frac{\partial}{\partial\hat\lambda}k_{0,q}+2\hat\lambda_{ll}\frac{\partial^2}{\partial\hat\lambda\partial\hat\lambda_{ll}}k_{0,q}+4\hat\lambda_{aabb}\frac{\partial^2 k_{0,q}}{\partial\hat\lambda\partial\hat\lambda_{aabb}}. \tag{49}$$

This relation, by using eq. (42) becomes

$$\begin{aligned} 0 &= (3q+4)\frac{\partial^q}{\partial\hat\lambda_{ll}^{\frac{q-1}{2}}\partial\hat\lambda_{aabb}^{\frac{q+1}{2}}}\frac{\partial}{\partial\hat\lambda}k_{0,0}+ \\ &+ 2\hat\lambda_{ll}\frac{\partial^{q+1}}{\partial\hat\lambda_{ll}^{\frac{q-1}{2}+1}\partial\hat\lambda_{aabb}^{\frac{q+1}{2}}}\frac{\partial}{\partial\hat\lambda}k_{0,0}+4\hat\lambda_{aabb}\frac{\partial^q}{\partial\hat\lambda_{ll}^{\frac{q-1}{2}}\partial\hat\lambda_{aabb}^{\frac{q+1}{2}}}\frac{\partial^2 k_{0,0}}{\partial\hat\lambda_{ccgg}\partial\hat\lambda}. \end{aligned}$$

We note that if this relation holds until a fixed q, taking its derivative with respect to $\hat\lambda_{ll}$ and then with respect to $\hat\lambda_{aabb}$, we obtain that it holds also with q+2 replacing q (q must be odd). Therefore, it suffices to impose this relation for the lower odd value of q, i.e q=1. In this case it becomes

$$0=7\frac{\partial^2}{\partial\hat\lambda\partial\hat\lambda_{ppqq}}k_{0,0}+2\hat\lambda_{ll}\frac{\partial^3}{\partial\hat\lambda\partial\hat\lambda_{ll}\partial\hat\lambda_{ppqq}}k_{0,0}+4\hat\lambda_{ppqq}\frac{\partial^3 k_{0,0}}{\partial\hat\lambda\partial\hat\lambda_{ppqq}^2},$$

which is a consequence of (48) because it is its second derivative with respect to $\hat\lambda$ and $\hat\lambda_{ppqq}$. In this way, we have seen that the conditions (14) and (15) give only the restriction (48) for $k_{0,0}$ and many identities.
So we have that every element of the matrix $k_{p,q}$ can be expressed as function of $k_{0,0}$ and this is restricted only by eqs. (43) and (48).
Let us conclude by exploiting these conditions and let us do it by using the expansion of $k_{0,0}$ around the state with $\hat\lambda_{ppqq}=0$, i.e.,

$$k_{0,0}=\sum_{s=0}^{\infty}\frac{1}{s!}k_s(\hat\lambda,\hat\lambda_{ll})\hat\lambda_{ppqq}^s. \tag{50}$$

Using (50), eq. (43) becomes

$$9\frac{\partial^2 k_s}{\partial\hat\lambda_{ll}^2}=\frac{\partial k_{s+1}}{\partial\hat\lambda}, \tag{51}$$

while eq. (48) transforms into

$$0=3\sum_{s=0}^{\infty}\frac{1}{s!}k_s\hat\lambda_{ppqq}^s+2\hat\lambda_{ll}\sum_{s=0}^{\infty}\frac{1}{s!}\frac{\partial k_s}{\partial\hat\lambda_{ll}}\hat\lambda_{ppqq}^s+4\sum_{s=1}^{\infty}\frac{1}{(s-1)!}k_s\hat\lambda_{ppqq}^s$$

i.e. $$\begin{cases} 3k_0 + 2\hat{\lambda}_{ll}\frac{\partial k_0}{\partial \hat{\lambda}_{ll}} = 0 & \text{for s=0,} \\ 3k_s + 2\hat{\lambda}_{ll}\frac{\partial k_s}{\partial \hat{\lambda}_{ll}} + 4sk_s = 0 & \text{for } s \geq 1; \end{cases}$$

but the relation for s=0 is contained in the other equation, so that they can be written as

$$(3+4s)k_s + 2\hat{\lambda}_{ll}\frac{\partial k_s}{\partial \hat{\lambda}_{ll}} = 0 \quad \forall s \geq 0,$$

whose solution is

$$k_s = \hat{\lambda}_{ll}^{-\frac{3+4s}{2}}\widetilde{k}_s(\hat{\lambda}). \tag{52}$$

This allows to rewrite eq. (51) as

$$\frac{\partial \widetilde{k}_{s+1}}{\partial \hat{\lambda}} = \widetilde{k}_s \frac{9}{4}(3+4s)(5+4s). \tag{53}$$

In this way we have found that $\widetilde{k}_0(\hat{\lambda})$ is an arbitrary single-variable function, while the other functions $\widetilde{k}_{s+1}(\hat{\lambda})$ are determined by (53), except for a numerable family of constants arising from integration.

5. The 13 moments model as a subsystem of the 14 moments one

To verify that the 13 moments case is a subsystem of the 14 moments one we will show that the relations obtained in [5] for the scalar functions $j_{0,q}$ are satisfied by the value of $k_{0,q}$ found here but considering $\hat{\lambda}_{ppqq} = 0$. Firstly we have to rewrite the expressions of $k_{0,q}$. Substituting eq. (50) into eq. (41) we have

$$\begin{aligned} k_{0,q} &= 3^{\frac{q}{2}}\frac{1}{q+1}\sum_{s=0}^{\infty}\frac{1}{s!}\frac{\partial^{\frac{q}{2}}k_s}{\partial \hat{\lambda}_{ll}^{\frac{q}{2}}}\frac{\partial^{\frac{q}{2}}\hat{\lambda}_{ppqq}^s}{\partial \hat{\lambda}_{ppqq}^{\frac{q}{2}}} = \\ &= 3^{\frac{q}{2}}\frac{1}{q+1}\sum_{s=\frac{q}{2}}^{\infty}\frac{1}{s!}\frac{\partial^{\frac{q}{2}}k_s}{\partial \hat{\lambda}_{ll}^{\frac{q}{2}}}s(s-1)\cdots(s-\frac{q}{2}+1)\hat{\lambda}_{ppqq}^{s-\frac{q}{2}} \end{aligned}$$

If we calculate this for $\hat{\lambda}_{ppqq} = 0$, only the term for $s = \frac{q}{2}$ remains, so our relations becomes

$$k_{0,q} = 3^{\frac{q}{2}}\frac{1}{q+1}\frac{\partial^{\frac{q}{2}}k_{\frac{q}{2}}}{\partial \hat{\lambda}_{ll}^{\frac{q}{2}}} \quad \text{with q even.}$$

Substituting eq. (50) into eq. (42), still making the previous considerations, we have

$$k_{0,q} = 3^{\frac{q-1}{2}}\frac{1}{q+2}\frac{\partial^{\frac{q-1}{2}}k_{\frac{q+1}{2}}}{\partial \hat{\lambda}_{ll}^{\frac{q-1}{2}}} \quad \text{with q odd.}$$

Now using eqs. (52) we obtain

$$k_{0,q} = \begin{cases} 3^{\frac{q}{2}}\frac{1}{q+1}\left(-\frac{1}{2}\right)^{\frac{q}{2}}\eta(3+2q,3q+1)\hat{\lambda}_{ll}^{-\frac{3+3q}{2}}\tilde{k}_{\frac{q}{2}} & \text{for q even,} \\ \\ 3^{\frac{q-1}{2}}\frac{1}{q+2}\left(-\frac{1}{2}\right)^{\frac{q-1}{2}}\eta(5+2q,3q+2)\hat{\lambda}_{ll}^{-\frac{4+3q}{2}}\tilde{k}_{\frac{q+1}{2}} & \text{for q odd.} \end{cases} \tag{54}$$

where $\eta(a,b) = a(a-2)(a-4)\cdots(b+2)b$.
Comparing this result with the corresponding one for $j_{0,q}$ in [5] (i.e. eqs. (56) and (57)), we find that they are the same, except for identifying

$$I_q(\hat{\lambda}) = \left(-\frac{3}{2}\right)^{\frac{q}{2}} \frac{1}{q+1}\eta(3+2q, 3q+1)\widetilde{k}_{\frac{q}{2}}(\hat{\lambda}) \tag{55}$$

and for setting $c_q = 0$.
It is easy to verify that with I_q given by eq. (55), the condition (58) of ref. [5] becomes exactly the present eq. (53), except for substituting q=2s+2, and viceversa. All the other results of ref. [5], for the 13 moments model, can be obtained by substituting $\hat{\lambda}_{aabb} = 0$ in the present ones except for the new restriction $c_q = 0$.
In other words, in ref. [5] the solution was found except for two families of constants, one arising from integration of eq. (58) in ref. [5] and another constituted by the constants c_q appearing in eq. (57). This second family of constants doesn't appear if the 13 moments model is obtained as a subsystem of the 14 moments one.

6. The comparison with the kinetic approach

The solution of our conditions proposed by the kinetic approach, see [2] and [9], is

$$\begin{aligned} h' &= \int F(\lambda + \lambda_i c^i + \lambda_{ij} c^i c^j + \lambda_{ill} c^i c^2 + \lambda_{aabb} c^4) dc_1 dc_2 dc_3 \\ \phi'^k &= \int F(\lambda + \lambda_i c^i + \lambda_{ij} c^i c^j + \lambda_{ill} c^i c^2 + \lambda_{aabb} c^4) c^k dc_1 dc_2 dc_3, \end{aligned}$$

(where F is related with the distribution function at equilibrium), and it is easy to see that it satisfies the conditions (12), (14), (15). We can now see that it is a particular case of our general solution. In fact eqs. (17) and (20) now become

$$\begin{aligned} \phi^{ii_1\cdots i_p j_1 \cdots j_q k_1 h_1 \cdots k_r h_r}_{p,q,r} &= \int F^{(p+q+r)}(\lambda + \frac{1}{3}\lambda_{ll}c^2 + \lambda_{aabb}c^4) \\ & \quad c^i c^{i_1} \cdots c^{i_p} c^{j_1} \cdots c^{j_q} c^{2q} c^{h_1} c^{k_1} \cdots c^{h_r} c^{k_r} dc_1 dc_2 dc_3, \\ h^{i_1\cdots i_p j_1 \cdots j_q k_1 h_1 \cdots k_r h_r}_{p,q,r} &= \int F^{(p+q+r)}(\lambda + \frac{1}{3}\lambda_{ll}c^2 + \lambda_{aabb}c^4) \\ & \quad c^{i_1} \cdots c^{i_p} c^{j_1} \cdots c^{j_q} c^{2q} c^{h_1} c^{k_1} \cdots c^{h_r} c^{k_r} dc_1 dc_2 dc_3, \end{aligned}$$

and it is easy to see that eqs. (22) and (23) are satisfied.
Eqs. (18) and (21) hold with

$$\phi_{p,q,r} = \frac{4\pi}{p+q+2r+2}\int_0^\infty F^{(p+q+r)}(\lambda + \frac{1}{3}\lambda_{ll}c^2 + \lambda_{aabb}c^4)c^{p+3q+2r+3}dc,$$
$$h_{p,q,r} = \frac{4\pi}{p+q+2r+1}\int_0^\infty F^{(p+q+r)}(\lambda + \frac{1}{3}\lambda_{ll}c^2 + \lambda_{aabb}c^4)c^{p+3q+2r+2}dc.$$

The eqs. (25) and (27) are consequences of these. The definitions (28) now become

$$\begin{aligned} k_{p,q} &= \frac{4\pi}{p+q+1}\int_0^\infty F^{(p+q)}(\lambda + \frac{1}{3}\lambda_{ll}c^2 + \lambda_{aabb}c^4)c^{p+3q+2}dc \text{if p+q is even,} \\ k_{p,q} &= \frac{4\pi}{p+q+2}\int_0^\infty F^{(p+q)}(\lambda + \frac{1}{3}\lambda_{ll}c^2 + \lambda_{aabb}c^4)c^{p+3q+3}dc \text{if p+q is odd.} \end{aligned}$$

From these it follows

$$k_{0,0} = 4\pi \int_0^\infty F(\lambda + \frac{1}{3}\lambda_{ll} c^2 + \lambda_{aabb} c^4) c^2 dc$$

and it is not difficult to see that eqs. (43) and (44) are satisfied.
Proof of eq. (48) needs an integration by parts, as follows

$$\begin{aligned} 0 &= 3\int_0^\infty F c^2 dc + \frac{2}{3}\lambda_{ll}\int_0^\infty F' c^4 dc + 4\lambda_{aabb}\int_0^\infty F' c^6 dc = \\ &= 3\int_0^\infty F c^2 dc + \int_0^\infty \left(\frac{dF}{dc}\right) c^3 dc = 3\int_0^\infty F c^2 dc + \left|F c^3\right|_0^\infty - \int_0^\infty 3F c^2 dc \end{aligned}$$

which is satisfied because

$$\lim_{c\to\infty} F c^3 = 0.$$

We can now see that eq. (50) holds with

$$k_s = 4\pi \int_0^\infty F^{(s)}(\lambda + \frac{1}{3}\lambda_{ll} c^2) c^{4s+2} dc,$$

of which eq. (51) is an easy consequence.
By using the change of the integration variables $c = \eta\lambda_{ll}^{-\frac{1}{2}}$, we obtain eq. (52) with

$$\widetilde{k}_s = 4\pi \int_0^\infty F^{(s)}(\lambda + \frac{1}{3}\eta^2)\eta^{4s+2} d\eta. \tag{56}$$

Proof of eq. (52) needs two integrations by part, as follows

$$\begin{aligned} \frac{d}{d\lambda}\widetilde{k}_{s+1} &= 4\pi \int_0^\infty F^{(s+2)}(\lambda + \frac{1}{3}\eta^2)\eta^{4s+6} d\eta = \\ &= \left| 4\pi F^{(s+1)}(\lambda + \frac{1}{3}\eta^2)\frac{3}{2}\eta^{4s+5} \right|_0^\infty + \\ &- \int_0^\infty 6\pi(4s+5)F^{(s+1)}(\lambda + \frac{1}{3}\eta^2)\eta^{4s+4} d\eta = \\ &= \left| -6\pi(4s+5)F^{(s)}(\lambda + \frac{1}{3}\eta^2)\frac{3}{2}\eta^{4s+3} \right|_0^\infty + \\ &- \int_0^\infty -9\pi(4s+5)(4s+3)F^{(s)}(\lambda + \frac{1}{3}\eta^2)\eta^{4s+2} d\eta = \\ &= \frac{9}{4}(4s+3)(4s+5)\widetilde{k}_s. \end{aligned}$$

Consequently, the kinetic approach suggest to take

$$\widetilde{k}_0(\lambda) = 4\pi \int_0^\infty F(\lambda + \frac{1}{3}\eta^2)\eta^2 d\eta,$$

which is only a change from our arbitrary function $\widetilde{k}_0(\lambda)$ to the arbitrary function F; moreover it considers only a particular solution of the eqs. (53), i.e., eq. (56). In this way the numerable family of arbitrary constants arising from integration of eq. (53) doesn't appear in the kinetic approach. Then the macroscopic approach here considered is more general than the kinetic one.

7. Conclusions

We are very satisfied by the present results because we have found the closure of the field equations up to whatever order with respect to equilibrium. This was never obtained before in literature. Apparently difficulties in calculation becomes very elegant and somehow simpler. Moreover, our model inherits all the nice properties of the Extended Thermodynamics, such as to be expressed in the form of a symmetric hyperbolic system, to predict finite speeds of propagation for shock waves, and to guarantee well posedness of the Cauchy problem and continuous dependence on the initial data.

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